Pythagoras' Theorem on a 2D-Lattice from a "Natural" Dirac Operator and Connes' Distance Formula

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Abstract

One of the key ingredients of A. Connes' noncommutative geometry is a generalized Dirac operator which induces a metric(Connes' distance) on the state space. We generalize such a Dirac operator devised by A. Dimakis *et al* , whose Connes' distance recovers the linear distance on a 1D lattice, into 2D lattice. This Dirac operator being "naturally" defined has the "local eigenvalue property" and induces Euclidean distance on this 2D lattice. This kind of Dirac operator can be generalized into any higher dimensional lattices.

Key words: Dirac operator, Noncommutative geometry, distance, lattice

I Introduction

Lattice Dirac operator is a long-standing problem embarrassing lattice field theorists. No-Go theorem [1] makes the implementation of chiral fermions on lattices almost impossible. Recent years, some breakthroughs have been achieved, e.g. the rediscovery of Ginsparg-Wilson relation [2], the devices of domainwall and overlap Dirac operators [3][4]. On the other hand, lattices can be considered as a simplest realization of noncommutative geometry (NCG) which has drawn more and more attention of theoretical physicists due to its applications in standard model of particle physics [5][6][7], lattice field theory [8][9][10], and string/M-theory [11][12]. NCG provides a powerful candidate of mathematical framework for geometrical understanding of fundamental physical laws. In Alain Connes's version of NCG, a generalized Dirac operator plays a central role in introducing the metric structure onto a noncommutative space [13][14][15]. Then intuitively, it becomes an interesting question whether Connes' NCG idea could brighten the problem of lattice Dirac operator. In fact, some groups have explored this question. G. Bimonte *et al* first pointed out that naïve Dirac operator is not able to induce the conventional distance on a 4D lattice by Connes' construction and Wilson-Dirac operator gives an even worse result [16]. Starting from this observation, E. Atzmon computed this "anomalous distance" induced by naïve Dirac operator for an 1D lattice precisely [17], which gives

$$d(0,2n-1)=2n, d(0,2n)=2\sqrt{n(n+1)}(n\in\mathcal{N})$$

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Hence, we can say that NCG provides another criterion for the rational choices of Dirac operator on lattices. A. Dimakis *et al* discovered an 1D operator whose Connes' distance coincides with the usual linear distance, when the number of lattice sites is finite [18]. This serial work indicates that it is a highly nontrivial problem to devise a proper Dirac operator whose Connes' distance is a desired one.

In this paper, we construct a "natural" Dirac operator on a 2D lattice. The induced metric of this operator endows conventional Euclidean geometry on this lattice, i.e. the Pythagoras' Theorem holds for the Connes' distance. The restoration of metric relies on that this operator has a so-called "local eigenvalue property", such that the norm $\| [D, f] \|$ is completely solvable (diagonalizable, integrable). The Dirac operator devised by us can be regarded as a generalization of Dimakis' operator in 1D case; it can be generalized into any high dimensional lattices easily.

The paper is organized as follows. In section II, we give a brief introduction to Connes' NCG and his distance formula. The "natural" Dirac operator is defined in section III. In section IV, we explore the "local eigenvalue property" of our Dirac operator. In section V, we prove that our Dirac operator implies the Pythagoras' Theorem on a 2D Lattice. In section VI, we generalize the "natural" Dirac operator into any higher dimensional lattices and make some open discussions.

II Noncommutative Geometry and Connes' Distance Formula

The object of a Connes' NCG is a triple (A, H, D) called a K-cycle, in which A is an involutive algebra, represented faithfully and unitarily as a subalgebra of bounded operators on a Hilbert space H, and D is a self-adjoint operator on H, which is called generalized Dirac operator, with compact resolvent so that $[D, \hat{a}]$ is bounded for all a in A. Here we use $\hat{a}$ to denote the imagine of a on H; without introducing any misunderstanding below, we will just omit the hat on a. A K-cycle is required to satisfy some axioms such that it recovers the ordinary spin geometry on a differential manifold when A is taken to be the algebra of smooth functions over this spin manifold, H is the space of L^2-spinors and D is the classical Dirac operator[15].

From a K-cycle, we can define a metric $d_D(,)$ on the state space of A denoted as $S(A)$, which we have referred as induced metric or Connes' distance.

$$d_D(\phi, \psi) = sup\{|\phi(a) - \psi(a)|/a \in A, \| [D, a] \| \leq 1\} \tag{1}$$

for any $\phi, \psi \in S(A)$. One can check that this definition satisfies all of the three axioms for a metric easily. Once A is commutative, pure states correspond to characters which can be interpreted as points, with A being the algebra of functions over these points (Gel'fand-Naimark Theorem); and Eq.(1) can, under this circumstance, be rewritten as

$$d_D(p, q) = sup\{|f(p) - f(q)|/f \in A, \| [D, f] \| \leq 1\}, \forall p, q \in S(A) \tag{2}$$

Noticing the fact that

$$\| a \|^2 = sup\{(a\psi, a\psi)/\psi \in H, \| \psi \| = 1\} = sup\{\lambda(a) \mid a^* a\psi = \lambda(a)\psi\}, a \in B(H)$$

where $B(H)$ is the algebra of bounded operators on H, we can express the inequality constraint in Eq.(2) by using eigenvalues

$$d_D(p, q) = sup\{|f(p) - f(q)|/f \in A, \lambda([D, f]^\dagger [D, f]) \leq 1\}, \forall p, q \in S(A) \tag{3}$$

Notations
$\sigma_i, i = 1, 2, 3$ are Pauli matrices defined in the ordinary way. If S is a finite set, then $|S|$ is the number of the elements in S. Let $i = 1, 2, 3$, define γ-matrices as

$$\gamma^i = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix}; \gamma^4 = \begin{pmatrix} & i \\ -i & \end{pmatrix}$$

The vacant matrix elements are understood as zeros and this will be taken as a convention all through this paper. Introduce $\gamma^1_\pm = \frac{1}{2}(\gamma^1 \pm i\gamma^2), \gamma^2_\pm = \frac{1}{2}(\gamma^3 \pm i\gamma^4)$, whose explicit matrix representations are

$$\gamma^1_+ = \begin{pmatrix} & & & 1 \\ & & 0 & \\ & 1 & & \\ 0 & & & \end{pmatrix}; \gamma^1_- = \begin{pmatrix} & & & 0 \\ & & 1 & \\ & 0 & & \\ 1 & & & \end{pmatrix};$$

$$\gamma^2_+ = \begin{pmatrix} & & 0 & \\ & & & -1 \\ 1 & & & \\ & 0 & & \end{pmatrix}; \gamma^2_- = \begin{pmatrix} & & 1 & \\ & & & 0 \\ 0 & & & \\ & -1 & & \end{pmatrix}$$

satisfying Clifford algebra relations

$$\{\gamma^i_\pm, \gamma^j_\pm\} = 0, \{\gamma^i_\pm, \gamma^j_\mp\} = \delta^{ij}, i, j = 1, 2$$

$\mathcal{Z}$ is referred to the set of integrals and $\mathcal{C}$ is for complex numbers. We adopt the convention $a = 1$ for the lattice constant a in this paper.

III "Natural" Dirac Operator on a 2D Lattice

First we give a detailed re-formulation of Dimakis' operator on the 1D lattice. Let $L_1 = \{x | x \in \mathcal{Z}\}$, $A_0 = \{f : L_1 \to \mathcal{C}\}, H = l^2(L_1) \otimes \mathcal{C}^2 = \{\psi\}$. The operator developed by Dimakis *et al* acting on H can be written in the form

$$D = \sigma_+\partial^+ + \sigma_-\partial^- = \begin{pmatrix} & \partial^+ \\ \partial^- & \end{pmatrix}$$

where $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$, $(T^\pm\psi)(x) := \psi(x \pm 1)$, $(\partial^\pm\psi)(x) = ((T^\pm - \mathbf{1})\psi)(x) = \psi(x \pm 1) - \psi(x)$ and $\psi(x) = (\psi_1, \psi_2)^T(x)$. Refine A_0 to be $A = \{f \in A_0 / \parallel [D, f] \parallel < \infty\}$.

We point out that D has a "local eigenvalue property", showing below

$$[D, f] = \begin{pmatrix} & (\partial^+ f)T^+ \\ (\partial^- f)T^- & \end{pmatrix} = -\begin{pmatrix} & T^+ \cdot (\partial^- f) \\ T^- \cdot (\partial^+ f) & \end{pmatrix}$$

$$[D, f]^\dagger = -[D, \bar{f}]$$

Introduce a "Hamiltonian" $H(df) := [D, f]^\dagger [D, f]$

$$= \begin{pmatrix} |\partial^+ f|^2 & \\ & |\partial^- f|^2 \end{pmatrix}$$

Consider the eigenvalue equation

$$H(df)\psi = \lambda(df)\psi \Leftrightarrow$$
$$|\partial^+ f|^2(x)\psi_1(x) = \lambda\psi_1(x), \forall x \in \mathcal{Z}$$
$$|\partial^- f|^2(x)\psi_2(x) = \lambda\psi_2(x), \forall x \in \mathcal{Z}$$

So that the constraint in Eq.(3) implies

$$\lambda(x,1) = |\partial^+ f|^2(x) \leq 1, \forall x \in \mathcal{Z} \tag{4}$$

$$\lambda(x,2) = |\partial^- f|^2(x) \leq 1, \forall x \in \mathcal{Z} \tag{5}$$

Now we observe that each eigenvalue of $H(df)$ is just related to one link of L_1, to which we refer as "local eigenvalue". As a comparison, the eigenvalue equation for 1D naïve Dirac operator $D_N := (T^+ - T^-)/2$ is

$$\frac{1}{4}(|\partial^+ f|^2(x) + |\partial^- f|^2(x))\psi(x) + \frac{1}{4}((\partial^+ \bar{f})(x)(\partial^+ f)(x+1)\psi(x+2) + (\partial^- \bar{f})(x)(\partial^- f)(x-1)\psi(x-2)) = \lambda\psi(x)$$

for all x in $\mathcal{Z}$, with $A_N = A$, $H_N = l^2(L_1)$, which possesses no "local eigenvalues" evidently.

Now we induce $d_D(,)$ on L_1. Let $f \in A$ subjected to (4)(5), i.e. $\parallel [D,f] \parallel \leq 1$, so that $|f((x+m) - f(x)| \leq m, m = 1,2,3,...$; hence, $d_D(x+m,x)$ has an upper bound m. Let $f_0(x) = x$, then f_0 saturates this upper bound. Thus, we have proved $d_D(m,n) = |m-n|, \forall m,n \in L_1$.

The 2D lattice being considered as a set is parametrized as $L_2 = \{x = (m,n) \mid m,n \in \mathcal{Z}\}$.

$$A_0 := \{f : L_2 \to \mathcal{C}\}, H := l^2(L_2) \otimes \mathcal{C}^4 = \{\psi : \psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T\}$$

$$D = \sum_{i=1}^{2}\sum_{s=\pm} \gamma_s^i \partial_i^s = \begin{pmatrix} & & \partial_2^- & \partial_1^+ \\ & & \partial_1^- & -\partial_2^+ \\ \partial_2^+ & \partial_1^+ & & \\ \partial_1^- & -\partial_2^- & & \end{pmatrix} \tag{6}$$

$$(T_i^{\pm}\psi)(x) = \psi(x \pm \hat{i})$$
$$(\partial_i^{\pm}\psi)(x) = ((T_i^{\pm} - \mathbf{1})\psi)(x) = \psi(x \pm \hat{i}) - \psi(x)$$

where $\hat{1} = (1,0), \hat{2} = (0,1)$. Refine A_0 to be $A := \{f \in A_0 / \parallel [D,f] \parallel < \infty\}$. By symmetries of D, we have

$$d_D((m,n),(m',n')) = d_D((0,0),(m'-m,n'-n))$$

so that to consider $d_D((0,0),(m,n))$ is enough. What's more,

$$d_D((0,0),(m,n)) = d_D((0,0),(-n,m)), d_D((0,0),(m,n)) = d_D((0,0),(n,m)),$$

so we just need to consider $m \geq n \geq 0$.

To end this section , we adopt the method in [16] to prove

$$d_D((0,0),(i,0)) = i, i = 1,2,... \tag{7}$$

Lemma 1 *Let $f \in A$, $\| [D,f] \| \leq 1$ and $\tilde{f}(m,n) := f(m,0)$, then $\| [D,\tilde{f}] \| \leq 1$.*

Proof:
By the definition of D in (6), there is

$$[D,\tilde{f}]\psi = ((\partial_1^+\tilde{f})(T_1^+\psi_4), (\partial_1^-\tilde{f})(T_1^-\psi_3), (\partial_1^+\tilde{f})(T_1^+\psi_2), (\partial_1^-\tilde{f})(T_1^-\psi_1))^T$$

Thus, notice the definition of $\tilde{f}$,

$$([D,\tilde{f}]\psi, [D,\tilde{f}]\psi) = \sum_{m,n}(|\partial_1^+\tilde{f}|^2(m,n)(\sum_{i=1}^{4}|\psi_i'|^2(m,n)))$$

$$= \sum_m(|\partial_1^+ f|^2(m,0)(\sum_n\sum_{i=1}^{4}|\psi_i'|^2(m,n)))$$

$$\Rightarrow \| [D,\tilde{f}] \| = sup\{|\partial_1^+ f|(m,0)/m \in \mathcal{Z}\} < \infty \tag{8}$$

Define $\hat{H} := \{\psi \in H/\psi_i = 0, i = 1,2,3\}$, then

$$\| [D,f] \|_{\hat{H}} \leq \| [D,f] \| \leq 1 \tag{9}$$

and

$$([D,f]\psi, [D,f]\psi)|_{\hat{H}} = \sum_{m,n}|\partial_1^+ f|^2(m,n)|\psi_4|^2(m+1,n) + |\partial_2^+ f|^2(m,n)|\psi_4|^2(m,n+1)$$

For any $\psi \in H$, we define $\hat{\psi} \in \hat{H}$ by that $\hat{\psi}_4(m,n) = 0, n \neq 0$ and that $\hat{\psi}_4(m,0)$ satisfy $|\hat{\psi}_4(m,0)|^2 = \sum_n\sum_{i=1}^4|\psi_i'|^2(m,n)$, for all m. Therefore,

$$([D,\tilde{f}]\psi, [D,\tilde{f}]\psi) = \sum_m(|\partial_1^+ f|^2(m,0)|\hat{\psi}_4(m,0)|^2 \leq ([D,f]\hat{\psi}, [D,f]\hat{\psi})|_{\hat{H}} \leq \| [D,f] \|_{\hat{H}}$$

Notice (9), and there are

$$\| [D,\tilde{f}] \| \leq \| [D,f] \|_{\hat{H}} \leq \| [D,f] \| \leq 1$$

$\square$

Following this lemma, we can just consider f with $\partial_2^+ f = 0$ and reach the conclusion $d_D((0,0),(i,0)) \leq i, i = 1,2,...;$ $f_0(m,n) = m$ saturates this upper bound. Hence, Eq.(7) holds.

IV Local Eigenvalues

We have to consider the eigenvalue problem in Eq.(3), when discussing distance $d_D((0,0),(m,n)), m \geq n \geq 1$. Fortunately, our "natural" Dirac operator has the "local eigenvalue property" also. Here we give a very detailed calculation.

Let $D(f) := [D,f]$

$$= \begin{pmatrix} & & (\partial_2^- f)T_2^- & (\partial_1^+ f)T_1^+ \\ & & (\partial_1^- f)T_1^- & -(\partial_2^+ f)T_2^+ \\ (\partial_2^+ f)T_2^+ & (\partial_1^+ f)T_1^+ & & \\ (\partial_1^- f)T_1^- & -(\partial_2^- f)T_2^- & & \end{pmatrix}$$

One can check $[D,f]^\dagger = -D(\bar{f})$. Define the "Hamiltonian" $H(df) := [D,f]^\dagger[D,f] = -D(\bar{f})D(f)$

$$= \begin{pmatrix} |\partial_1^+ f|^2 + |\partial_2^- f|^2 & (T_2^-(\partial_2^+ \bar{f}\partial_1^+ f) - T_1^+(\partial_1^- \bar{f}\partial_2^- f))T_1^+T_2^- \\ (T_1^-(\partial_1^+ \bar{f}\partial_2^+ f) - T_2^+(\partial_2^- \bar{f}\partial_1^- f))T_1^-T_2^+ & |\partial_1^- f|^2 + |\partial_2^+ f|^2 \end{pmatrix} \oplus$$

$$\begin{pmatrix} |\partial_1^+ f|^2 + |\partial_2^+ f|^2 & (T_2^+(\partial_2^- \bar{f}\partial_1^+ f) - T_1^+(\partial_1^- \bar{f}\partial_2^+ f))T_1^+T_2^+ \\ (T_1^-(\partial_1^+ \bar{f}\partial_2^- f) - T_2^-(\partial_2^+ \bar{f}\partial_1^- f))T_1^-T_2^- & |\partial_1^- f|^2 + |\partial_2^- f|^2 \end{pmatrix}$$

with the eigenvalue equation

$$H(df)\psi = \lambda(df)\psi \tag{10}$$

Fortunately, Eq.(10) can be reduced to a collection of equation sets, with each equation set being related to a fundamental plaque $\{(m,n),(m+1,n),(m,n-1),(m+1,n-1)\} \subset L_2$

$$\begin{pmatrix} \rho_1^2 + \rho_4^2 - \lambda & \overline{\Delta_4}\Delta_3 - \overline{\Delta_1}\Delta_2 \\ \Delta_4\overline{\Delta_3} - \Delta_1\overline{\Delta_2} & \rho_2^2 + \rho_3^2 - \lambda \end{pmatrix} \begin{pmatrix} \psi_1(m,n) \\ \psi_2(m+1,n-1) \end{pmatrix} = 0 \tag{11}$$

$$\begin{pmatrix} \rho_3^2 + \rho_4^2 - \lambda & -\overline{\Delta_4}\Delta_1 + \overline{\Delta_3}\Delta_2 \\ -\Delta_4\overline{\Delta_1} + \Delta_3\overline{\Delta_2} & \rho_1^2 + \rho_2^2 - \lambda \end{pmatrix} \begin{pmatrix} \psi_3(m,n-1) \\ \psi_4(m+1,n) \end{pmatrix} = 0 \tag{12}$$

in which $\Delta_1(m,n) := (\partial_1^+ f)(m,n)$, $\Delta_2(m,n) := (\partial_2^+ f)(m+1,n-1)$, $\Delta_3(m,n) := (\partial_1^+ f)(m,n-1)$, $\Delta_4(m,n) := (\partial_2^+ f)(m,n-1)$ and, without misleading, we have omitted the arguments (m,n). Let $\Delta_i := \rho_i e^{i\theta_i}, i = 1,2,3,4$, $\theta := \theta_1 + \theta_3 - \theta_2 - \theta_4$, and

$$A := \rho_1^2 + \rho_4^2, B := \rho_2^2 + \rho_3^2, C := \overline{\Delta_4}\Delta_3 - \overline{\Delta_1}\Delta_2$$

$$A' := \rho_3^2 + \rho_4^2, B' := \rho_1^2 + \rho_2^2, C' := -\overline{\Delta_4}\Delta_1 + \overline{\Delta_3}\Delta_2$$

then the secular equation for Eq.(11) is

$$(\lambda - A)(\lambda - B) - C\bar{C} = \lambda^2 - (A+B)\lambda + AB - C\bar{C} = 0$$

in which

$$C\bar{C} = \rho_3^2\rho_4^2 + \rho_1^2\rho_2^2 - 2\rho_1\rho_2\rho_3\rho_4\cos\theta$$

The solutions are

$$\lambda_\pm = \frac{1}{2}(A + B \pm \sqrt{(A-B)^2 + 4C\bar{C}}) \tag{13}$$

Similarly for Eq.(12),

$$\lambda'_\pm = \frac{1}{2}(A' + B' \pm \sqrt{(A'-B')^2 + 4C'\bar{C}'}) \tag{14}$$

in which

$$C'\bar{C}' = \rho_1^2\rho_4^2 + \rho_2^2\rho_3^2 - 2\rho_1\rho_2\rho_3\rho_4\cos\theta$$

Therefore, $\parallel [D,f] \parallel \leq 1 \Leftrightarrow \forall(m,n)$,

$$\lambda_+(m,n) \leq 1, \lambda'_+(m,n) \leq 1 \tag{15}$$

Equivalently, Eqs.(13)(14) and inequality (15) $\Leftrightarrow$

$$1 + \rho_1^2\rho_3^2 + \rho_2^2\rho_4^2 + 2\rho_1\rho_2\rho_3\rho_4\cos\theta \geq \rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2 \tag{16}$$

$$2 \geq \rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2 \tag{17}$$

There is another constraint, the closedness condition

$$\Delta_1 + \Delta_4 = \Delta_2 + \Delta_3 \tag{18}$$

Hence we get (16)(17)(18) as the specific expressions of $\| [D, f] \| \leq 1$ on L_2.

To end this section, we give Eq.(7) a second proof using (15). In fact, just notice

$$\rho_1^2 \leq \rho_1^2 + \rho_4^2 \leq \lambda_+ \leq 1$$

and we have $\rho_i \leq 1, i = 1, 2, 3, 4$. Consequently, $d_D((0,0),(i,0))$ has the upper bound i.

V Pythagoras' Theorem for $d_D(,)$ on L_2

Let $m \geq n \geq 1$ below and we claim that

Theorem 1 *(Pythagoras' Theorem on 2D lattice)*

$$d_D((0,0),(p,q)) = \sqrt{p^2+q^2}, \forall p \geq q \geq 1 \tag{19}$$

The proof needs only inequalities (17)(18) and we will treat $q = 1$ and $q = 2, 3, 4, ...$ respectively. We need an important inequality in mathematics analysis

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \leq n \sum_{i=1}^{n} a_i b_i \tag{20}$$

where the equality holds if $a_1 = a_2 = ... = a_n$ or $b_1 = b_2 = ... = b_n$.

The general philosophy of the proof can be illustrated as following. First we define the concept of *canonical path*, which means a subset of L_2 generated by one point denoted as *start* and the operations T_1^+, T_2^+. Then it must have a *end* if T_1^+, T_2^+ just act finite times. Second we pick out a rectangle subset $L(p,q)$ of L_2 which is defined as $\{(m,n)/m = 0, 1, ..., p, n = 0, 1, .., q\}$, and define the link set on $L(p,q)$, $B(p,q) := \{(m,n,s)/s = 1, 2; s = 1 : m = 0, 1, ..., p-1, n = 0, 1, ..., q; s = 2 : m = 0, 1, ..., p, n = 0, 1, ..., q-1\}$. For any $f|_{L(p,q)}$ defined on $L(p,q)$, there is a function Δf defined on $B(p,q)$ correspondingly, with $\Delta f(m,n,1) = (\partial_1^+ f)(m,n), \Delta f(m,n,2) = (\partial_2^+ f)(m,n)$. There is another set induced by $L(p,q)$, the set of all canonical paths starting at $(0,0)$ and ending at (p,q), written as $\Gamma(p,q)$. Any element $\gamma \in \Gamma(p,q)$ can be considered as $p+q$ sequential links, namely a subset of $B(p,q)$. As for the first step of proof, we want to show for any $\gamma \in \Gamma(p,q)$, $|\sum_{l\in\gamma} \Delta f(l)| \leq \sqrt{p^2+q^2}$, if f subjects to (17)(18). Adopting reduction of absurdity, we suppose there is a γ_0 such that $|\sum_{l\in\gamma_0} \Delta f(l)| > \sqrt{p^2+q^2}$. Then by virtue of the closedness condition (18), $|\sum_{l\in\gamma} \Delta f(l)| > \sqrt{p^2+q^2}, \forall \gamma \in \Gamma(p,q)$, which implies

$$\sum_{l\in\gamma} |\Delta f(l)|^2 > \frac{p^2+q^2}{p+q}$$

by (20). Therefore

$$\sum_{\gamma\in\Gamma(p,q)} (\sum_{l\in\gamma} \Delta f(l)^2) > R(p,q) \frac{p^2+q^2}{p+q} \tag{21}$$

where we introduce $R(p,q) = \sum_{\gamma \in \Gamma(p,q)} 1 = |\Gamma(p,q)|$, the number of all canonical paths starting from $(0,0)$ and ending (p,q), which is equal to the binomial coefficient $\begin{pmatrix} p+q \\ p \end{pmatrix}$. We change the summation for paths in $\Gamma(p,q)$ in (21) to the summation in $B(p,q)$,

$$\sum_{l \in B(p,q)} (L(l)\Delta f(l)^2) > R(p,q)\frac{p^2+q^2}{p+q} \tag{22}$$

where $L(l)$ is the weight function on $B(p,q)$ introduced by the changing of indice of summations. The geometric interpretation of $L(l)$ is the number of canonical paths in $\Gamma(p,q)$ passing through link l, which is expressed as

$$L(m,n,1) = R(m,n)R(p-m-1,q-n), L(m,n,2) = R(m,n)R(p-m,q-n-1)$$

We induce the third set from $L(p,q)$, the set $Q(p,q)$ of all fundamental plaques as subsets in $L(p,q)$, which can be expressed as $\{(m,n)/m = 0,1,...,p-1, n = 0,1,...,q-1\}$. Rewrite (17) as

$$\sum_{i=1}^{4} |\Delta f|^2(p,i) \leq 2 \tag{23}$$

which can be considered as a constraint on the four links in one fundamental plaque p. Now sum (23) for all plaques in $Q(p,q)$

$$\sum_{p \in Q(p,q)} \sum_{i=1}^{4} |\Delta f|^2(p,i) \leq 2|Q(p,q)| = 2pq$$

Again we change the summation for plaques to the summation for links and introduce another weight function $S(l)$,

$$\sum_{l \in B(p,q)} S(l)|\Delta f|^2(l) \leq 2|Q(p,q)| \tag{24}$$

where $S(m,0,1) = S(m,q,1) = S(0,n,2) = S(p,n,2) = 1, S(other) = 2$, namely to links l shared by two plaques $S(l) = 2$, else $S(l) = 1$. Using (20) again and noticing $L(l) \neq 0$, we get

$$2|Q(p,q)| \geq \sum_{l \in B(p,q)} S(l)|\Delta f|^2(l) \geq \sum_{l \in B(p,q)} (\frac{S(l)}{L(l)})(L(l)|\Delta f|^2(l))$$

$$\geq \frac{1}{|B(p,q)|} \sum_{l \in B(p,q)} (\frac{S(l)}{L(l)}) \sum_{l \in B(p,q)} (L(l)|\Delta f|^2(l))$$

$$\Leftrightarrow \frac{2|Q(p,q)||B(p,q)|}{\sum_{l \in B(p,q)} \frac{S(l)}{L(l)}} \geq \sum_{l \in B(p,q)} (L(l)|\Delta f|^2(l)) \tag{25}$$

where $|B(p,q)| = 2pq + p + q$. The contradiction lays between inequalities (22) and (25), if only

Lemma 2

$$\frac{2|Q(p,q)||B(p,q)|}{\sum_{l \in B(p,q)} \frac{S(l)}{L(l)}} \leq R(p,q)\frac{p^2+q^2}{p+q} \tag{26}$$

As we declared that the description above is just general philosophy of proof, since the operating on $L(p,q)$ is complicated, we will just prove $q=1$ case in Lemma 2 and leave $q\geq 2$ cases to a revised $L(p,q)$ construction.

As for $q=1$, (26)$\Leftrightarrow$

$$2p(1+3p)\leq(1+p^2)\sum_{l\in B(p,1)}\frac{S(l)}{L(l)} \tag{27}$$

Using the definitions of $L(l)$ and $S(l)$, we have $\sum_{l\in B(p,1)}\frac{S(l)}{L(l)}=2(p+\sum_{n=1}^{p}\frac{1}{n})$. Then (27) reduces to show

$$3p^2\leq p^3+(1+p)(\sum_{n=1}^{p}\frac{1}{n}) \tag{28}$$

However, (28) is easily checked by $p=1$, $p=2$, and $p=3,4,...$ respectively. Accordingly, we reach the conclusion $d_D((0,0),(p,1)),p=1,2,3,...$ has an upper bound $\sqrt{1+p^2}$.

We define $\tilde{L}(p,q)$ to be a "folding ruler", $\{(m,0),(m,1),(p-1,n),(p,n)/m=0,1,..,p,n=2,...,q\}$, for $p\geq q\geq 2$, and apply our general philosophy of proof to $\tilde{L}(p,q)$. Namely, we induce $\tilde{B}(p,q),\tilde{P}(p,q),\tilde{Q}(p,q)$ from $\tilde{L}(p,q)$, and introduce weight functions $\tilde{L}(l),\tilde{S}(l)$ on $\tilde{B}(p,q)$. If

Lemma 3

$$\frac{2|\tilde{Q}(p,q)||\tilde{B}(p,q)|}{\sum_{l\in\tilde{B}(p,q)}\frac{\tilde{S}(l)}{\tilde{L}(l)}}\leq\tilde{R}(p,q)\frac{p^2+q^2}{p+q} \tag{29}$$

holds, then we find a contradiction and conclude that $d_D((0,0),(p,q))$ has an upper bound $\sqrt{p^2+q^2}$ for all $p\geq q\geq 2$. And here, $|\tilde{Q}(p,q)|=p+q-1$, $|\tilde{B}(p,q)|=3p+3q-2$, $\tilde{R}(p,q)=1+pq$; $\tilde{S}(m,0,2)=\tilde{S}(p-1,n,1)=2,m=1,2,...,p-1,n=1,...,q-1,\tilde{S}(other)=1$; $\tilde{L}(m,0,1)=1+q(p-m-1),\tilde{L}(m,0,2)=q,m=0,1,...p-1$; $\tilde{L}(p,0,2)=1$; $\tilde{L}(m,1,1)=q(m+1),m=0,1,...p-2$; $\tilde{L}(p-1,n,2)=p(q-n),\tilde{L}(p,n,2)=1+pn,n=1,...,q-1$; $\tilde{L}(p-1,n,1)=p,n=1,...,q$.

Thus,

$$\sum_{l\in\tilde{B}(p,q)}\frac{\tilde{S}(l)}{\tilde{L}(l)}=2+2(\frac{p}{q}+\frac{q}{p})+\delta\geq 6$$

in which

$$\delta=\frac{1}{q}(\sum_{n=1}^{p-1}\frac{1}{n}-1)+\frac{1}{p}(\sum_{n=1}^{q-1}\frac{1}{n}-1)+\sum_{n=1}^{p-1}\frac{1}{1+nq}+\sum_{n=1}^{q-1}\frac{1}{1+np}\geq 0$$

We claim Lemma 3 is implied by

$$(p+q)|\tilde{Q}(p,q)||\tilde{B}(p,q)|\leq 3\tilde{R}(p,q)(p^2+q^2)$$

$$\Leftrightarrow (p+q)(p+q-1)(3(p+q)-2)\leq 3(1+pq)(p^2+q^2),\forall p\geq q\geq 2 \tag{30}$$

We prove (30) by induction. First, fixing one $q=2,3,...$, we show that (30) holds when $p=q$. In this case, (30) is reduced to

$$12q^3+2q\leq 3q^4+13q^2,\forall q\geq 2$$

The above inequality can be checked to be valid when $q = 2$, $q = 2$, and $q \geq 3$. Second, with this fixed q, we suppose that the statement is valid in the case $p = p_0$, and show that it will also hold in $p = p_0 + 1$. Without misunderstanding, we drop the subscript "0". It is sufficient to check

$$9pq + 9q^2 \leq 9(q-1)p^2 + 3q^3 + 7p + 4q + 3$$

Since $9pq \leq 9(q-1)p^2, \forall p \geq q \geq 2$, we just check

$$9q^2 \leq 3q^3 + 7p + 4q + 3$$

which is valid when $q = 2$ and $q \geq 3$. So the upper bound $\sqrt{p^2 + q^2}$ follows.

We choose $f_{(p,q)}(m,n) = \frac{pm+qn}{\sqrt{p^2+q^2}}$ which can be checked easily to saturate this upper bound. Therefore, theorem 1 follows.

VI Discussions

Our "natural" Dirac operator is able to be generalized easily into any dimension greater than two. Let $\Gamma^i, i = 1, 2, ..., 2d$ be the generators of $Cl(E^{2d})$ which satisfy

$$\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}, i, j = 1, 2, ..., 2d \tag{31}$$

Define $\Gamma^k_\pm = (\Gamma^{2k-1} \pm i\Gamma^{2k})/2, k = 1, 2, ..., d$, and (31) changes the form into

$$\{\Gamma^m_\pm, \Gamma^n_\pm\} = 0, \{\Gamma^m_\pm, \Gamma^n_\mp\} = \delta^{mn}, m, n = 1, 2, ..., d$$

Now let

$$D_{(k)} = \sum_{k=1}^{d} \sum_{s=\pm} \Gamma^k_s \partial^s_k$$

and one can check the "square root property"

$$(D_{(k)})^2 = \sum_{k=1}^{d} \partial^+_k \partial^-_k$$

B. Iochum *et al* gave a confirmative answer to the question whether there exists a Dirac operator which gives a desired metric on a finite space in [19]. For infinite case, as we have mentioned, the sequential works [16][17][18] together with ours show that this problem is highly nontrivial. Besides, work on the implications of our "natural" Dirac operator on physics is in proceeding.

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